Solving the Heat Equation Using Separation of Variables

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Abstract: **The heat equation is one of the fundamental partial differential equations in mathematical physics, describing how heat diffuses through a medium over time. It appears in various fields, such as thermodynamics, fluid dynamics, and materials science, and is a cornerstone of understanding diffusive processes. Mathematically, the heat equation in one spatial dimension is expressed as:**

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x} \ ,$$

where $(u(x, t)$ represents the temperature at position (x) and time (t), and (α) is the thermal diffusivity, a positive constant that depends on the material properties. Solving this equation analytically can be challenging due to its dependence on both space and time, but one of the most elegant and widely applicable methods is separation of variables. This technique reduces the PDE into a set of ordinary differential equations (ODEs), which are easier to solve. In this article, we will explore the separation of variables method in detail, including its assumptions, step-by-step derivation, and a practical example applied to a finite rod with fixed boundary conditions.

Keywords: **heat equation, thermodynamics, fluid dynamics, variables method.**

1. UNDERSTANDING THE HEAT EQUATION AND ITS PHYSICAL CONTEXT

Before delving into the solution method, it's worth understanding the physical significance of the heat equation. The equation models the diffusion of heat in a medium, such as a metal rod, where the rate of change of temperature with respect to time $\frac{\partial u}{\partial t}$ is proportional to the spatial curvature of the temperature distribution $\frac{\partial^2 u}{\partial x}$ The constant α encapsulates properties like thermal conductivity, density, and specific heat capacity, making the equation adaptable to different materials.

To solve the heat equation, we must specify initial and boundary conditions. For instance, consider a rod of length (L) where the ends are held at a fixed temperature ($u(0, t) = 0$ and $u(L, t) = 0$) and an initial temperature distribution $u(x, 0) = f(x)$ is given. These conditions define a well-posed problem that the separation of variables method can address effectively.

2. THE SEPARATION OF VARIABLES METHOD: CONCEPTUAL OVERVIEW

The core idea of separation of variables is to assume that the solution $(u(x, t))$ can be written as a product of two functions: one depending only on space, $(X(x))$, and the other depending only on time, $(T(t))$. That is:

$u(x, t) = X(x)\,T(t).$

This assumption simplifies the PDE by separating the spatial and temporal variables into independent equations. If this form holds, substituting it into the heat equation and manipulating the result leads to two ODEs whose solutions can be combined to satisfy the original problem's conditions. The method relies on the linearity of the heat equation and the specific form of the boundary conditions, making it particularly suited for problems with homogeneous boundary conditions (e.g., zero temperature at the boundaries).

Step-by-Step Derivation

Let's apply the separation of variables method to the one-dimensional heat equation with boundary conditions $u(0,t) = 0, u(L,t) = 0$, and initial condition $u(x,0) = f(x)$.

Substitute the Assumed Form:

Substitute $u(x,t) = X(x)\,T(t)$ into the heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x}$$

Compute the derivatives:

- The time derivative: $\frac{\partial u}{\partial t} = X(x)\frac{d}{t}$.
- The spatial second derivative: $\frac{\partial^2 u}{\partial t x^2} = \alpha T(t)\frac{dX}{dx^2}$

Plugging these into the heat equation gives:

$$X(x)\frac{dT}{dt} = \alpha T(t)\frac{d^2X}{dx^2}$$

1. **Separate Variables:**

Divide both sides by $(X(x)\,T(t))$,

assuming $X(x) \neq 0$ and $T(t) \neq 0$:

$$\frac{1\,dT}{T(t)dt} = \frac{\alpha d^2X}{X(x)dx^2}$$

The left-hand side depends only on (t), and the right-hand side depends only on (x). For this equality to hold for all (x) and (t), both sides must equal a constant, say $-\lambda$:

$$\frac{1\,dT}{T(t)dt} = \frac{\alpha d^2X}{X(x)dx^2} = -\lambda$$

The negative sign is chosen for convenience, as it leads to physically meaningful solutions (exponential decay in time).

2. **Form Two ODEs:**

This gives two separate equations:

- Time equation: $\frac{dT}{dt} = -\lambda T(t)$
- Spatial equation: $\frac{d^2X}{dx^2} = -\frac{\lambda}{\alpha}\,X(x)$

3. **Solve the Spatial Equation with Boundary Conditions:**

The spatial equation is:

$$\frac{d^2X}{dx^2} + \frac{\lambda}{\alpha}\,x(x) = 0$$

This is a second-order ODE with the general solution:

$$X(x) = Acos\left(\frac{\sqrt{\lambda}}{\alpha}x\right) + Bsin\left(\frac{\sqrt{\lambda}}{\alpha}x\right)$$

where (A) and (B) are constants. Apply the boundary conditions:

- $u(0,t) = X(0)\,T(t) = 0 \Rightarrow X(0) = 0$
- $u(L,t) = X(L)\,T(t) = 0 \Rightarrow X(L) = 0$
- For $X(0) = 0$:

$$X(0) = A\cos(0) + B\sin(0) = A = 0 \Rightarrow A = 0.$$

So, $X(x) = B\sin\left(\frac{\sqrt{\lambda}}{\alpha}x\right) = 0$

For $X(L) = 0$:

$$X(L) = B\sin\left(\frac{\sqrt{\lambda}}{\alpha}L\right) = 0$$

Since $B \neq 0$ (otherwise $X(x) = 0$, a trivial solution), we require:

$\sin\left(\frac{\sqrt{\lambda}}{\alpha}L\right) = 0.$

This holds when $\frac{\sqrt{\lambda}}{\alpha}L = n\pi$, where $n = 1, 2, 3, \ldots$

(positive integers ensure non-trivial solutions). Solving for λ:

$\frac{\sqrt{\lambda}}{\alpha} = \frac{n\pi}{L} \Longrightarrow \frac{\lambda}{\alpha} = (\frac{n\pi}{L})^2 \Longrightarrow \lambda_n = \alpha(\frac{n\pi}{L})^2$

Thus, the spatial solutions are:

$$X_n(x) = B_n\sin\left(\frac{n\pi x}{L}\right)$$

Where $B_n(x)$is a constant for each (n).

4. Solve the Time Equation:

The time equation is:

$$\frac{dT}{dt} = -\lambda_n T = -\alpha(\frac{n\pi}{L})^2 T$$

The solution is:

$$T_n(t) = C_n e^{-\alpha(\frac{n\pi}{L})^2 T}$$

where C_n is a constant.

5. Construct the General Solution:

For each (n), the solution is:

$$u_n(x,t) = X_n(x)T_n(t) = B_n C_n e^{-\alpha(\frac{n\pi}{L})^2 t}\sin\left(\frac{n\pi x}{L}\right)$$

Let $a_n = B_n C_n$. Since the heat equation is linear, the general solution is a superposition:

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\alpha(\frac{n\pi}{L})^2 t}\sin\left(\frac{n\pi x}{L}\right)$$

6. Apply the Initial Condition:

At $t = 0, u(x,0) = f(x)$:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x).$$

This is a Fourier sine series. The coefficients a_n are determined by:

$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx,$

using the orthogonality of the sine functions over $([0, L])$.

3. EXAMPLE APPLICATION: COOLING OF A ROD

Consider a rod of length $L = 1$ with $\alpha = 1$,

initial temperature $f(x) = \sin(\pi x)$, and boundary conditions

$u(0,t) = 0, u(1,t) = 0$.

Compute the coefficients:

$a_n = 2\int_0^1 sin(\pi x)\sin(n\pi x)dx$.

Using orthogonality, $a_1 = 1$ $(when\ n = 1)$ and $a_n = 0\ for\ n \neq 1$. Thus:

$u(x,t) = e^{-\pi^2 t} sin(\pi x)$.

This solution shows the temperature decaying exponentially while maintaining a sinusoidal spatial profile, consistent with heat diffusion.

4. ADVANTAGES AND LIMITATIONS

The separation of variables method is powerful for problems with regular geometries and homogeneous boundary conditions. However, it struggles with non-homogeneous conditions or irregular domains, where numerical methods or transform techniques (Laplace or Fourier transforms) might be more appropriate.

5. CONCLUSION

The separation of variables method provides an elegant, analytical approach to solving the heat equation, transforming a complex PDE into manageable ODEs. Its reliance on Fourier series connects it to broader mathematical theory, making it a cornerstone of applied mathematics. Through this method, we gain insight into the dynamic process of heat diffusion, applicable to countless real-world scenarios.

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